



Research Signpost
37/661 (2), Fort P.O.
Trivandrum-695 023
Kerala, India

Recent Developments in Wear Prevention, Friction and Lubrication, 2010: 1-43
ISBN: 978-81-308-0377-7 Editor: George K. Nikas

1. The thin film approximation in hydrodynamic, including elastohydrodynamic, lubrication

Andras Z. Szeri

Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, USA

Abstract. The equations that describe the motion of viscous fluids are difficult to solve even with present day computing facilities and we constantly search for ways to simplify them. A notable simplification that arises in lubrication is the so-called “thin film” or “lubrication” approximation. The resulting Reynolds theory of lubrication is a constant viscosity, quasi two-dimensional theory, valid when the ratio of the characteristic lengths is vanishingly small. It breaks down where there is a sudden change in film thickness, or if the Reynolds number is increased even with the flow remaining laminar. Among the additional circumstances that negate validity of the classical Reynolds theory is the viscosity being strongly dependent on the pressure or on another component of stress. The Reynolds theory also breaks down if the film becomes too thin for the continuum model to remain applicable.

Introduction

Osborne Reynolds developed the thin film approximation in his efforts to explain the experimental results of Beauchamp Tower. While studying Tower’s report on railroad bearings, Reynolds identified “crucial proof ... that the surfaces were completely and continuously separated by a film of oil; this film being maintained by the motion of the journal, although the pressure in the oil at the crown of the bearing was shown by actual measurement to be as much as 625 lbs. per sq. inch above the pressure in the oil bath” [1]. It further occurred to Reynolds as possible that “the film of oil might be sufficiently thick for the unknown boundary actions to disappear, in which case the results would be deducible from the equations of hydrodynamics”. During the course of his research, Reynolds linearized the Navier-Stokes equations for flow between slightly inclined, rigid surfaces. The resulting Reynolds theory of lubrication is a constant viscosity, quasi two-dimensional theory, valid when the ratio of the characteristic lengths is

vanishingly small. It breaks down where there is a sudden change in film thickness, or if the Reynolds number is increased even with the flow remaining laminar. Among the additional circumstances that negate validity of the classical Reynolds theory is the viscosity being strongly dependent on the pressure or on another component of stress. The Reynolds theory also breaks down if the film becomes too thin for the continuum model to remain applicable.

1. The Reynolds equation

Reynolds based his theory of lubrication on the following assumptions [1].

1. The continuum description is valid.
2. The Navier-Stokes equations hold (viscosity depends at most on temperature).
3. The lubricant is incompressible.
4. The film is thin, therefore
 - (a) lubricant flow is laminar;
 - (b) lubricant inertia is negligible;
 - (c) lubricant film curvature is negligible;
 - (d) lubricant body force is negligible.

According to Reynolds, as consequence of assumption 4(c) it is permissible to describe fluid film lubrication relative to orthogonal Cartesian coordinates. Conventionally, the y -axis of the Cartesian system is in the direction of the minimum film dimension while the ‘plane’ of the lubricant film coincides with the (x,z) plane.

Applications of the above assumptions to the Navier-Stokes equations led Reynolds to the reduced equations of motion

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = \mu \frac{\partial^2 w}{\partial y^2} \quad (1)$$

The equation of continuity, in contrast, retained its original form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2)$$

The boundary condition assigned to these equations by Reynolds were no-slip on the solid surfaces

$$\begin{aligned} u &= U_1, \quad w = 0 \quad \text{at} \quad y = 0 \\ u &= U_2, \quad w = 0 \quad \text{at} \quad y = h \end{aligned} \tag{3}$$

To the present order of approximation, the pressure does not vary across the film (1) and the equations of motion could be integrated with respect to y

$$\begin{aligned} u &= \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - yh) + \left(1 - \frac{y}{h}\right) U_1 + \frac{y}{h} U_2, \\ w &= \frac{1}{2\mu} \frac{\partial p}{\partial z} (y^2 - yh) \end{aligned} \tag{4}$$

The pressure is, yet, unknown. However, since it is an induced pressure (Reynolds postulated ambient pressure at film's edges) with the sole function of enforcing conservation of mass, it can be evaluated from the equation of continuity. This seems like a reasonable scheme, but has one flaw. If u and w are substituted into equation (2) the resulting single equation will contain two unknowns, v and p , and unless v is given, we have insufficient information to determine p . Reynolds overcome the problem by integrating the equation of continuity across the film; the integrated equation of continuity contained the velocity component v only in the values it assumed at the boundaries $y = 0$ and $y = h(x, t)$. As the approach velocity of the surfaces was presumed known during this analysis, integration across the film eliminated one of the two unknowns.

Substituting for u and w into the equation of continuity and integrating across the film resulted in what we now call the *Reynolds equation of lubrication*

$$\frac{\partial}{\partial x} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial z} \right) = 6(U_1 - U_2) \frac{\partial h}{\partial x} + 6h \frac{\partial (U_1 + U_2)}{\partial x} + 12(V_2 - V_1). \tag{5}$$

Here $V_1 - V_2$ is the velocity of approach of the surfaces,

It is emphasized that $V_1 = (U_1, V_1)$ and $V_2 = (U_2, V_2)$ are the velocities of "corresponding" points, each fixed to one of the bearing surfaces.¹ The velocities V_1 and V_2 result from rigid body motion that may include both rotation and translation of the bearing surfaces. To take cognizance of this we recast the equation in a form that contains relative, rather than absolute, velocities (see, for e.g. [2])

¹ We call two points, one fixed to the bearing surface and the other to the runner surface, corresponding points at the instant when they are located on the same normal to the reference surface. For journal bearings, the pad surface is the reference surface. For the plane slider, on the other hand, it is expedient to designate the runner surface as the reference surface [2].

$$\frac{\partial}{\partial x} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial z} \right) = 6U_0 \frac{\partial h}{\partial x} + 6h \frac{\partial U_0}{\partial x} + 12V_0. \quad (6)$$

The interpretation put on the velocity U_0 is distinctly different in the two cases. For thrust bearings $U_0 = U_1 - U_2$, in contrast $U_0 = U_1 + U_2$ for journal bearings. In both cases, however, $V_0 = V_1 - V_2$.

Table 1 demonstrates convergence of the Sommerfeld number $S = \mu N(R/C)^2/P$ with the clearance ratio (C/R) for a journal bearing under Gumbel's boundary conditions [3].

In this Chapter, we will examine some of the consequences and restrictions that follow from the assumptions of Reynolds. In Section 2, we will examine the restrictions brought about by the thin-film assumption. In Section 3, we discuss behavior of piezoviscous fluids and Section 4 will highlight the lower limit in film thickness for molecularly thin films.

Table 1. Convergence of Sommerfeld number with the clearance ratio.

<i>Model</i>	<i>C/R</i>	<i>S</i>
Navier-Stokes	0.002	0.33692
	0.001	0.33698
	0.0005	0.33701
Reynolds lubrication	0.0	0.33704

2. The thin-film assumption for Navier-Stokes fluids

In this section, we investigate the thin film assumption for a Navier-Stokes fluid. The equations of motion and continuity governing the flow of constant property Newtonian fluid are

$$\rho \frac{dv_i}{dt} = \frac{\partial}{\partial x_j} \left(-p \delta_{ij} + 2\mu D_{ij} \right) \quad (7)$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

We normalize these equations with characteristic length scales L_{xz} , and L_y and characteristic velocity scales U_* and $V_* = (L_y/L_{xz})U_*$, along and across the film, respectively

$$\begin{aligned} (x_1, x_2, x_3) &= L_{xz} (X, \varepsilon Y, Z) \\ (v_1, v_2, v_3) &= U_* (U, \varepsilon V, W) \end{aligned} \quad (8)$$

Here $\varepsilon = L_y/L_{xz}$ is the ratio of characteristic lengths, in conventional lubricant films $\varepsilon = O(10^{-3})$. We normalize time and pressure according to

$$t = \left(\frac{L_{xz}}{U_*}\right) \tau, \quad p = \frac{\rho U_*^2}{Re^*} P \quad (9)$$

where

$$Re = \frac{U_* L_y}{\nu}, \quad \text{and} \quad Re^* = \varepsilon Re,$$

are the Reynolds number and the reduced Reynolds number, respectively. This choice for normalization leaves the continuity equation formally invariant and retains the pressure term in the limit $Re^* \rightarrow 0$.

2.1. Laminar flow

Recasting the Navier-Stokes equations in terms of normalized variables, one obtains [4]

$$-\varepsilon^2 \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} \right) + Re^* \frac{dU}{d\tau} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (10a)$$

$$\varepsilon^2 \left\{ -\varepsilon^2 \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Z^2} \right) + Re^* \frac{dV}{d\tau} - \frac{\partial^2 V}{\partial Y^2} \right\} = -\frac{\partial P}{\partial Y} \quad (10b)$$

$$-\varepsilon^2 \left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2} \right) + Re^* \frac{dW}{d\tau} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \quad (10c)$$

The system represented by equations (7₂) and (10) contains two parameters, the aspect ratio ε and the reduced Reynolds number Re^* . We will now investigate the significance of two asymptotic cases provided by limiting values of these parameters.

Case (A): $Re^* > \varepsilon^2 \rightarrow 0$

Neglecting terms multiplied by ε^2 equation (10) yields

$$Re^* \frac{dU}{d\tau} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (11a)$$

$$P = P(X, Z, \tau) \quad (11b)$$

$$Re^* \frac{dW}{d\tau} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \quad (11c)$$

The second of these equations states that the flow is a quasi two-dimensional Navier-Stokes flow: the pressure is invariant along L_y . Clearly, it is not possible to characterize this flow by a single equation in pressure. It is only the second approximation $Re^* \rightarrow 0$, that makes possible the derivation of a single pressure equation.

Case (B): $\varepsilon^2 > Re^* \rightarrow 0$

Neglecting terms multiplied by Re^* in equation (10) leads to a three-dimensional Stokes flow

$$\nabla^2 \mathbf{V} = -\nabla P \quad (12)$$

It is, again, not possible to characterize this flow by a single equation in pressure and the full three-dimensional problem must be solved [5, 6]. However, in lubrication this limit is of interest only in the most unusual of circumstances; the Reynolds number must approach zero faster than ε locally. For machined surfaces L_y might be equated to the asperity height, δ , and the lateral length scale L_{xz} to the distance between asperities, l ; for ground surfaces the local value of the aspect

ratio is then $\sim 1.25\mu/12.5\mu$, or, $\varepsilon^2 \sim 0.01 > Re^*$, forcing the Reynolds number to be so small as to be outside interest in most lubrication application.

2.2. Turbulent flow

For sufficiently large Reynolds number assumption (4a) no longer holds as the flow might become turbulent even in thin lubricant films. In this case the equations of motion for the mean flow are obtained by substituting the assumptions

$$p = \bar{P} + p' \quad v_i = \bar{V}_i + v'_i$$

into equation (7₁) and averaging.

$$\rho \frac{d\bar{V}_i}{dt} = \frac{\partial}{\partial x_j} \left(-\bar{P} \delta_{ij} + 2\mu \bar{D}_{ij} - \overline{\rho v'_i v'_j} \right) \quad (13)$$

The over-score bar signifies the average value of the quantity and the prime its instantaneous departure from the average; the material derivative d/dt and the stretching tensor \bar{D}_{ij} both refer to mean quantities

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{V}_k \frac{\partial}{\partial x_k}, \quad \bar{D}_{ij} = \frac{1}{2} (\bar{V}_{ij} + \bar{V}_{ji})$$

The coordinates are normalized as for laminar flow (8₁) while the velocities and the pressure according to

$$\left\{ \begin{array}{l} (\bar{V}_1, \bar{V}_2, \bar{V}_3) = \frac{1}{U_*} (U, \varepsilon V, W), \quad \bar{P} = \frac{\rho U_*^2}{Re^*} P \\ (v'_1, v'_2, v'_3) = u_*(u, v, w), \quad \kappa = (u_*/U_*)^2 \end{array} \right. \quad (14)$$

Here we assume that the fluctuating components of the velocity are all of the same order of magnitude, $v'_1 \sim v'_2 \sim v'_3$ [7]. We have not yet specified the value of κ , the square of the ratio of characteristic velocities of the fluctuations and of the mean flow. In fact, we shall investigate the consequences of selecting for κ one value or another.

Substitution into equation (13) leads to the normalized form of the equations for turbulent flow

$$Re^* \left[\frac{dU}{d\tau} + \kappa \left(\frac{\partial \bar{u}u}{\partial X} + \frac{\partial \bar{u}w}{\partial Z} \right) \right] + \kappa Re \frac{\partial \bar{u}v}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} + \varepsilon^2 \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} \right) \quad (15a)$$

$$\varepsilon^2 \left[Re^* \frac{dV}{d\tau} - \frac{\partial^2 V}{\partial Y^2} - \varepsilon^2 \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Z^2} \right) \right] + \kappa \varepsilon Re^* \left(\frac{\partial \bar{v}u}{\partial X} + \frac{\partial \bar{v}w}{\partial Z} \right) = -\frac{\partial P}{\partial Y} - \kappa Re^* \left(\frac{\partial \bar{v}v}{\partial Y} \right) \quad (15b)$$

$$Re^* \left[\frac{dW}{d\tau} + \kappa \left(\frac{\partial \bar{u}w}{\partial X} + \frac{\partial \bar{w}w}{\partial Z} \right) \right] + \kappa Re \frac{\partial \bar{v}w}{\partial Y} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} + \varepsilon^2 \left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2} \right) \quad (15c)$$

These equations contain three independent parameters ε , Re^* or Re , and κ . Based on the previous discussion, we focus attention on the limit $Re^* > \varepsilon^2 \rightarrow 0$ while investigating the role of κ . Setting the condition $\varepsilon^2 = 0$ we obtain

$$Re^* \left[\frac{dU}{d\tau} + \kappa \left(\frac{\partial \bar{u}u}{\partial X} + \frac{\partial \bar{u}w}{\partial Z} \right) \right] + \kappa Re \frac{\partial \bar{u}v}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (16a)$$

$$\kappa Re^* \left[\varepsilon \left(\frac{\partial \bar{v}u}{\partial X} + \frac{\partial \bar{v}w}{\partial Z} \right) + \frac{\partial \bar{v}v}{\partial Y} \right] = -\frac{\partial P}{\partial Y} \quad (16b)$$

$$Re^* \left[\frac{dW}{d\tau} + \kappa \left(\frac{\partial \bar{u}w}{\partial X} + \frac{\partial \bar{w}w}{\partial Z} \right) \right] + \kappa Re \frac{\partial \bar{v}w}{\partial Y} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \quad (16c)$$

Most current turbulent lubrication theories that are based on a single equation for pressure [8-10], take their departure from equation (16) on the assumption that

$$\kappa = (u_*/U_*)^2 = O(1) \quad (17)$$

Hinze [7] shows, however, that this assumption is valid only in free turbulent flows such as jets and wakes, i.e., in flows that differ qualitatively from flows in confined, narrow spaces. Nevertheless, when inserting this value of κ into equation (16) one obtains

The thin film approximation in hydrodynamic, including elasto-hydrodynamic, lubrication

$$Re^* \left[\frac{dU}{d\tau} + \frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{u}}{\partial Z} \right] + Re \frac{\partial \bar{u}}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (18a)$$

$$Re^* \frac{\partial \bar{v}}{\partial Y} = -\frac{\partial P}{\partial Y} \quad (18b)$$

$$Re^* \left[\frac{dW}{d\tau} + \frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{w}}{\partial Z} \right] + Re \frac{\partial \bar{w}}{\partial Y} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \quad (18c)$$

Only under the highly non-physical assumption

$$Re^* \rightarrow 0, Re = O(1)$$

is it possible to combine equation (18) with the equation of continuity to yield a single equation in pressure

$$\frac{\partial}{\partial x} \left(\frac{h^3}{\mu k_x(Re_h)} \frac{\partial \bar{P}}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{\mu k_z(Re_h)} \frac{\partial \bar{P}}{\partial z} \right) = \frac{U_0}{2} \frac{\partial h}{\partial x} \quad (19)$$

Here $k_x = k_x(Re_h)$, $k_z = k_z(Re_h)$, and $Re_h(x) = [h(x)/L_y] Re$ stands for the local Reynolds number [2].

Constantinescu [11] attempted to demonstrate that the conditions leading up to equation (19) hold in typical lubrication situations. He did not address, however, the validity of the 'free turbulence' assumption (17).

Instead of using equation (17), one might recognize that due to the thinness of the lubricant film the bounding walls have controlling effect on the developing super laminar flow and, following Hinze [7], employ

$$\kappa = (u_*/U_*)^2 = O(\varepsilon) \quad (20)$$

Applying equation (20), equation (16) yields

$$Re^* \left(\frac{dU}{d\tau} + \frac{\partial \bar{u}}{\partial Y} \right) = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (21a)$$

$$P = P(X, Z, \tau) \quad (21b)$$

$$Re^* \left(\frac{dW}{d\tau} + \frac{\partial \overline{vw}}{\partial Y} \right) = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \quad (21c)$$

Although a Reynolds type pressure equation is not available for turbulent flow under assumption (20) either; however, in first approximation the pressure no longer varies across the film. Furthermore, for $Re^* \geq 1$ inertia terms are of the same order of magnitude as the turbulence terms: if there is no inertia, there can be no turbulence.

We may thus conclude that when classical lubrication assumptions no longer yield acceptable results in 'thin' flows, extension to quasi two-dimensional Navier-Stokes, rather than three-dimensional Stokes, equations should be made. The equations governing this flow are

$$\begin{aligned} Re^* \left(\frac{dU}{d\tau} + \alpha \frac{\partial \overline{uv}}{\partial Y} \right) &= -\frac{\partial P(X, Z, \tau)}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \\ Re^* \left(\frac{dW}{d\tau} + \alpha \frac{\partial \overline{vw}}{\partial Y} \right) &= -\frac{\partial P(X, Z, \tau)}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} &= 0 \end{aligned} \quad (22)$$

where $\alpha = 0$ for laminar flow and $\alpha = 1$ for turbulent flow.

2.3. The plane slider

We aim now to provide some evidence for the validity of the system in (22), at least in laminar flow. For infinite extent of the flow domain in the z-direction the steady laminar flow equation becomes quasi one-dimensional

$$\begin{aligned} Re^* \left(U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} \right) &= -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0 \end{aligned} \quad (23)$$

The results we shall quote here relate to flow between inclined, nominally flat planes, i.e., the ‘plane slider’. We put $h_1 = h(x_1)$ and $h_2 = h(x_2)$ for the film thickness at outlet and inlet, respectively, and specify the length scales by $L_y = (h_1 + h_2)/2$ and $L_{xz} = B = x_2 - x_1$, where x_1 and x_2 defines the position of the outlet and the inlet, respectively.

Introduction of the stream function $\Psi(X, Y)$ into (23) leads to [4]

$$Re^* \left(H \frac{\partial \Psi}{\partial \eta} \frac{\partial^3 \Psi}{\partial \eta^2 \partial \xi} - 2 \frac{dH}{dX} \frac{\partial \Psi}{\partial \eta} \frac{\partial^2 \Psi}{\partial \eta^2} - H \frac{\partial \Psi}{\partial \xi} \frac{\partial^3 \Psi}{\partial \eta^3} \right) - \frac{\partial^4 \Psi}{\partial \eta^4} = 0, \quad 0 \leq \xi, \eta \leq 1 \quad (24)$$

where we also introduced a change of variables

$$\begin{aligned} \xi &= X - X_1 \\ \eta &= Y / H(X), \quad H = h(x) / L_y \end{aligned}$$

The no-slip boundary conditions on the solid boundaries are

$$\begin{aligned} \Psi &= 0, \quad \frac{\partial \Psi}{\partial \eta} = -H, \quad \text{at } \eta = 0 \\ \Psi &= Q^*, \quad \frac{\partial \Psi}{\partial \eta} = 0, \quad \text{at } \eta = 1 \end{aligned} \quad (25)$$

where Q^* is the dimensionless flow rate, yet unknown. To insure that the problem remains mathematically well posed we increase the number of independent equations by constraining the average pressure at outlet to equal its value at inlet:

$$\int_0^1 \int_0^1 H(X) \frac{\partial P}{\partial \xi} d\xi d\eta = 0 \quad (26)$$

We approximate $\Psi(\xi, \eta)$ by piecewise polynomial functions [12] and apply Galerkin's method to evaluate the coefficients in the approximation. The resulting system of nonlinear algebraic equations can be written in the form

$$G(\omega) = 0, \quad \omega = (u, \sigma), \quad (27)$$

where u is the vector of state variables and σ is the vector of parameters. The computational scheme for solving equation (27), i.e., parametric continuation followed by the Gauss – Newton method, can be found in [4].

The principal conclusion from equation (22) is the invariance of the pressure across the film. To investigate the upper bound of ε for this conclusion to hold, we look at flow between inclined planes of various aspect ratios. As long as equations (22) hold, the pressure on the upper plate, $P(h)$, and the pressure on the lower plate, $P(0)$, are approximately equal, becoming identical at the limit $\varepsilon \rightarrow 0$. This may be investigated quantitatively by computing a pressure difference coefficient, d_p

$$d_p = |P(h) - P(0)|_{max} / P(h)_{max} \quad (28)$$

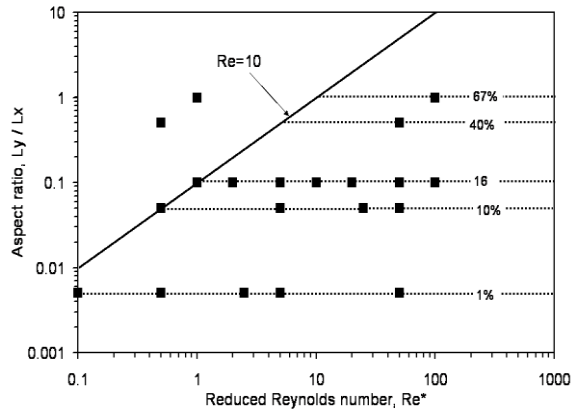


Figure 1. Pressure difference coefficient d_p for various values of the aspect ratio and reduced Reynolds number.

In figure 1, we indicate the value of d_p , calculated from the three-dimensional Navier-Stokes problem employing FIDAP, as a function of the parameters ε and Re^* . We restrict attention here arbitrarily to $Re > 10$, accepting this as a lower bound on the Reynolds number for applications. For 'small' values of the aspect ratio, figure 1 appears to support the assertion of equation (22): for $\varepsilon \leq 0.05$, $d_p \leq 0.01$, and even for the wider range $\varepsilon \leq 0.1$, $d_p < 0.16$, though the increase in d_p for $\varepsilon > 0.1$ is quite rapid. Thus, for $\varepsilon \leq 0.1$, we have the approximate relationship $d_p \approx (Re^*)$. This conclusion seems to hold well for $Re^* \leq 100$.

Figure 2 plots the ratio of actual pressure over its zero Reynolds number value against Re^* , as calculated by FIDAP from the three-dimensional Navier-Stokes problem at various values of $\varepsilon \leq 0.1$. Data for different ε values collapse onto a

single curve, confirming, again, that under the stated conditions the aspect ratio is not a strong parameter of the flow, that is $P_{max}/P_{0,max} \approx \Phi(Re^*)$.

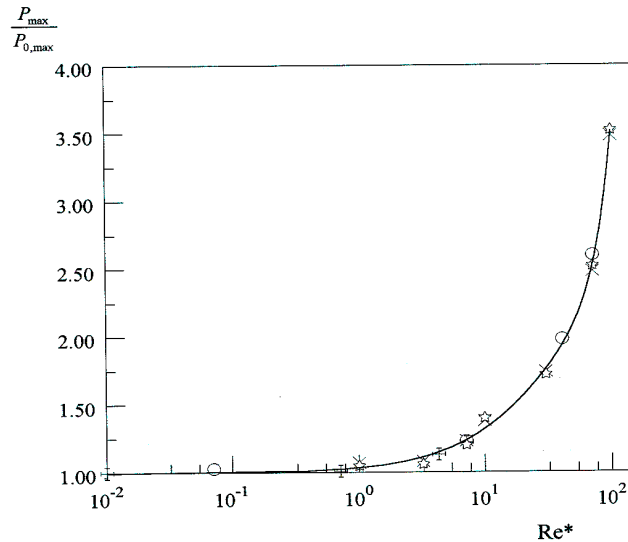


Figure 2. Variation of $P_{max}/P_{0,max}$ with Re^* , FIDAP (+, $\varepsilon = 0.005$; o, $\varepsilon = 0.05$; *, $\varepsilon = 0.08$; x, $\varepsilon = 0.1$).

Figure 3 compares lubricant force from two sources, FIDAP solution of the full Navier-Stokes problem and the stream function-Galerkin formulation of equations (24). In this plot the force is normalized with its zero Reynolds number value, and two channel geometries, $h_2/h_1 = 2$ and $h_2/h_1 = 3/2$ are depicted.

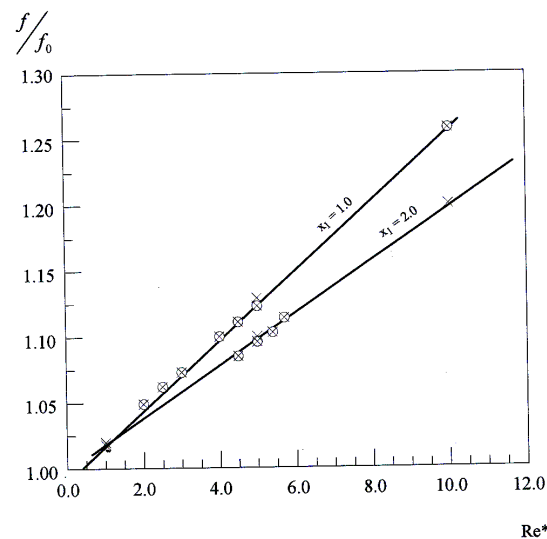


Figure 3. Normalized force, $\varepsilon \leq 0.1$ (o, approximation; x, Navier-Stokes).

The approximation (22) will now be used to investigate the effect of waviness of the runner. To this end, we perturb the film thickness to $H(\xi) = X_1 + \delta \cos(n\pi\xi)$. Figure 4 displays results for $n=5$, $\delta = \pm 0.1$ and $Re^* = 4.0$. When $\delta > 0$, the film shape is convergent in the direction of the flow at inlet and divergent at outlet, we characterize this as a c/d film shape. When $\delta < 0$, the film shape is d/c. It may be seen from figure 4 that for film shape c/d the effect of convective fluid inertia is to lower the pressure within the channel, while for film shape d/c the pressure is raised relative to inertialess flow. By changing to $n = 5.5$, figure 5, $\delta > 0$ yields a d/d film shape, resulting in the channel walls being pulled together. $\delta < 0$, on the other hand, establishes a c/c film and the channel walls are forced apart.

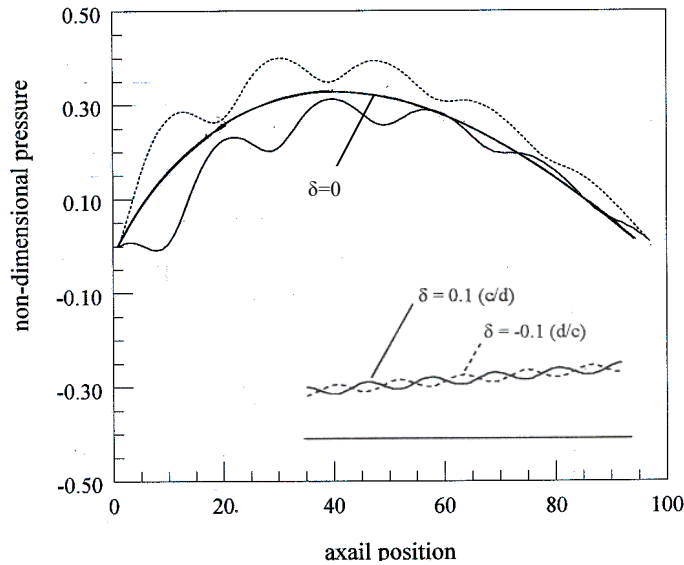


Figure 4. Perturbed slider ($n=5$, $X_1 = 2.0$, $\delta = \pm 0.1$, $Re^* = 4.0$) (— convergent at inlet, divergent at outlet; ---- divergent at inlet, convergent at outlet).

We suggest here that the system of equations (22) may constitute a logical first extension of the Reynolds thin film model. Provided that $\varepsilon \leq 0.1$, the normalized maximum pressure varies with the Reynolds number but is independent of the aspect ratio. Furthermore, the excess force due to convective inertia is additive for sinusoidal films convergent in the flow direction at exit [4]

$$\text{sgn}(f - f_0) = -\text{sgn} \left. \frac{dh}{dx} \right|_{x_{out}} \quad (29)$$

Here x is now increasing in the flow direction.

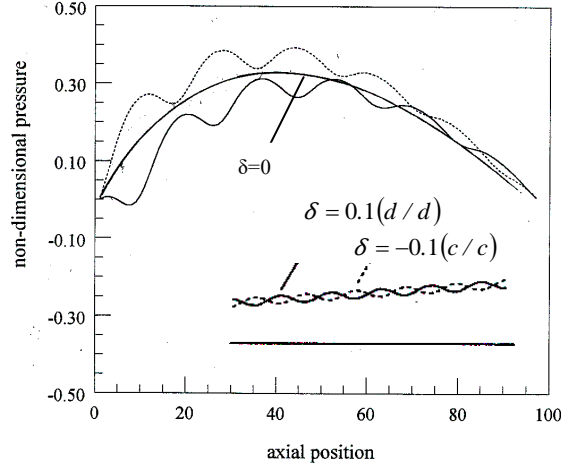


Figure 5. Perturbed slider ($n=5.5$ $X_1=2.0$, $\delta = \pm 0.1$, $Re^*=4.0$) (— convergent at inlet, divergent at outlet; ---- divergent at inlet, convergent at outlet).

2.4. Film curvature

To study the effect of curvature of the film (journal bearings) we employ a bipolar coordinate system $\{\hat{\alpha}, \hat{\beta}\}$ that is related to our Cartesian coordinate system $\{X, Y\}$ through

$$\hat{\alpha} + i\hat{\beta} = -2 \coth^{-1} \left(\frac{X + iY}{a} \right) \quad (30)$$

where a is the separation between the pole and the origin of the $\{X, Y\}$ system. In the bipolar coordinate system the cylinders of radii r_1 and r_2 , $r_1 < r_2$, have the simple representation $\hat{\alpha} = \hat{\alpha}_1$ and $\hat{\alpha} = \hat{\alpha}_2$, $\hat{\alpha}_1 < \hat{\alpha}_2 < 0$. The scale factor of the bipolar coordinate system [13] is $H = a / (\cosh \hat{\alpha} - \cos \hat{\beta})$.

The equations of motion and continuity defining the two-dimensional flow field are first written relative to the bipolar coordinate system [13] and then nondimensionalized [2]. In terms of bipolar coordinates the eccentricity ratio is given by

$$\varepsilon = \frac{\sinh(\hat{\alpha}_1 - \hat{\alpha}_2)}{\sinh \hat{\alpha}_1 - \sinh \hat{\alpha}_2} \quad (31)$$

The objective here is to find approximate solutions for thin films without neglecting film curvature and compare with Reynolds equation results. To discover

the correct approximation to use, we adopt the length scales $L_\beta = r_1$ along the principal dimension of the film and $L_\alpha = r_1/\Delta$, $\Delta = 1/(\hat{\alpha}_2 - \hat{\alpha}_1)$, across it. The characteristic velocities are $U_* = r_1\omega$ and $V_* = U_*/2\pi\Delta$. The dynamic condition in the flow are represented by the Reynolds number

$$r_e = \frac{U_* L_\alpha}{\nu}$$

Neglecting terms of the order $(\hat{\alpha}_2 - \hat{\alpha}_1)^2$ or smaller, the normalized equations of motion thus take the form

$$\begin{aligned} \frac{\partial P}{\partial \alpha} &= 0 \\ -\frac{1}{(2\pi)^2 H} \frac{\partial P}{\partial \beta} &= \frac{1}{H^2 \sinh \hat{\alpha}_1} \frac{\partial^2 v}{\partial \alpha^2} + r_e^* \left[\frac{1}{H} \left(u \frac{\partial v}{\partial \alpha} + v \frac{\partial v}{\partial \beta} \right) \right] \end{aligned} \quad (32)$$

Here u, v , are normalized velocities, $\alpha = (\hat{\alpha} - \hat{\alpha}_1)\Delta$ and $\beta = \hat{\beta}/2\pi$, and $r_e^* = r_e/2\pi\Delta$ is the reduced Reynolds number of the problem analogous to Re^* . The condition $Re^* \rightarrow 0$ of classical lubrication theory is equivalent, thus, to $r_e^* \rightarrow 0$ and upon applying this limit the equations reduces to

$$-\sinh \hat{\alpha}_1 \frac{H}{(2\pi)^2} \frac{\partial P}{\partial \beta} = \frac{\partial^2 v}{\partial \alpha^2}, \quad \frac{\partial P}{\partial \alpha} = 0 \quad (33)$$

To solve equation (33) we integrate twice with respect to α . Substitution into the integrated (across the film) continuity equation yields [3]

$$\begin{aligned} \frac{\partial}{\partial \beta} \left\{ \frac{\partial P}{\partial \beta} \int_0^1 H(\alpha, \beta) \left[\int_0^\alpha I(\sigma, \beta) d\sigma - \alpha \int_0^1 I(\sigma, \beta) d\sigma - C(1-\alpha) \right] d\alpha \right\} &= 0 \\ I(\alpha, \beta) &= \int_0^\alpha H(\varphi, \beta) d\varphi \end{aligned} \quad (34)$$

The innermost integral of equation (34) was obtained analytically while the other integrals necessary to solve for P were performed via Gaussian quadrature [3]. The error committed by neglecting curvature of the film maybe gauged from table 2, which displays values of the nondimensional group $P/\mu N$ under various conditions.

Table 2. Effect of film curvature on $(P/\mu N) \times 10^{-5}$.

Boundary condition	(C/R)	Navier-Stokes	Reynolds lub.	Bipolar lub.
Sommerfeld	0.002	7.4202	7.4175	7.4052
	0.001	29.6754	29.6701	29.6454
	0.0005	118.6908	118.6802	118.6732
Gümbel	0.002	14.8104	14.8052	14.7824
	0.001	59.2311	59.2207	59.1486
	0.0005	236.8966	236.8826	236.8125

The exact, zero Reynolds number solution in [14] valid for arbitrary clearance ratio, can be employed to study film curvature effects. When this solution is expanded in powers of the clearance ratio, the first two terms correspond to the Myllerup and Hamrock [15] solution, which employs regular perturbation:

$$\begin{aligned}
 P = & \frac{12\pi\varepsilon \sin \theta (2 + \varepsilon \cos \theta)}{(2 + \varepsilon^2)(1 + \varepsilon \cos \theta)^2} \\
 & + \left(\frac{C}{R}\right) \frac{4\pi\varepsilon \sin \theta (1 + 5\varepsilon^2 + 2\varepsilon(2 + \varepsilon^2)\cos \theta)}{(2 + \varepsilon^2)(1 + \varepsilon \cos \theta)^3} + O\left(\frac{C}{R}\right)^2
 \end{aligned} \tag{35}$$

The first term is identical to the solution of the Reynolds equation under full film boundary conditions, while the second term is the first order curvature correction [16].

3. Departure from Newtonian fluid behavior

The lubrication approximation has been widely used far outside the confines of lubrication, becoming one of the cornerstones of fluid mechanics [17-20]. It is important, therefore, to be cognizant of the assumptions under which it is derived, so that it is not used outside its range of applicability.

3.1. The Navier-Stokes model

The compressible Navier-Stokes equation assumes that the Cauchy stress \mathbf{T} depends only on the density and the velocity gradient.

$$\mathbf{T} = \mathbf{T}(\rho, \mathbf{L}), \quad \mathbf{L} = \text{grad} \mathbf{v} \quad (36)$$

The requirement of frame-indifference then implies that the velocity gradient occur only through its symmetric part. Restrictions due to isotropy and the assumption that the stress be linear in the symmetric part \mathbf{D} of the velocity gradient lead to the compressible Navier-Stokes model

$$\mathbf{T} = \alpha_0 (\rho, I_D, II_D, III_D) \mathbf{1} + \alpha_1 (\rho, I_D, II_D, III_D) \mathbf{D} + \alpha_2 (\rho, I_D, II_D, III_D) \mathbf{D}^2 \quad (37)$$

where

$$\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T),$$

$$I_D = \text{tr} \mathbf{D}, \quad II_D = \frac{1}{2} [(\text{tr} \mathbf{D})^2 - \text{tr} \mathbf{D}^2], \quad III_D = \det \mathbf{D}.$$

If the fluid is assumed to be incompressible, then it will follow (using the assumption that the constraint response does no work) that

$$\mathbf{T} = -p \mathbf{1} + \hat{\alpha}_1 (II_D, III_D) \mathbf{D} + \hat{\alpha}_2 (II_D, III_D) \mathbf{D}^2 \quad (38)$$

where p is the Lagrange multiplier due to the incompressibility constraint.

Requiring that the stress depend linearly on \mathbf{D} leads to

$$\mathbf{T} = -p \mathbf{1} + 2\hat{\mu} \mathbf{D}, \quad (39)$$

where $\hat{\mu}$ is a constant.

During his derivation of the model, Stokes [21] already recognized that the viscosity of a fluid could depend on the pressure, and he was concerned with the question as to when it would be reasonable to assume that the viscosity is a constant. “Let us now consider in what cases it is allowable to suppose μ to be independent of pressure. Du Buat has concluded it from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure... I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure” [21].

That the viscosity for liquids could depend on the pressure and could change significantly with sufficiently large variations of the pressure has been well

recognized [22]. Experiments that are more recent also confirm the fact that the viscosity of certain lubricants can change dramatically with the pressure (see [23-27]). Incorporating this notion into the development of constitutive theories leads to an interesting departure from classical theories.

Assuming that the constraint forces, in our case the pressure, can influence the work done, we have the possibility that the stress \mathbf{T} is of the form

$$\mathbf{T} = -p\mathbf{1} + \tilde{\alpha}_1(p, II_D, III_D)\mathbf{D} + \tilde{\alpha}_2(p, II_D, III_D)\mathbf{D}^2. \quad (40)$$

Further assuming linearity in \mathbf{D} , we arrive at

$$\mathbf{T} = -p\mathbf{1} + 2\mu(p)\mathbf{D}. \quad (41)$$

There is a very fundamental difference between the two models (39) and (41). While the model (39) provides an explicit relation between \mathbf{T} and \mathbf{D} , model (41) is implicit with the more general the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}. \quad (42)$$

A generalization of the model (42) in which the viscosity depends on both the pressure and the symmetric part of the velocity gradient permits one to describe shear thinning and shear-thickening, observed in some fluids. In such fluids the stress is of the form

$$\mathbf{T} = -p\mathbf{1} + \beta(p, \mathbf{D})\mathbf{D}. \quad (43)$$

While many models for viscoelastic fluids are implicit models, the popular Maxwell model being one such, they are not of the type (42); they usually involve higher derivatives of the stress and the symmetric part of the velocity gradient. Rigorous global existence of solutions for a sub-class of fluids of the form (43) in which the viscosity satisfies certain conditions, met by the models used in elastohydrodynamics, can be found in Malek *et al.* [28, 29]. Of course, for fluids of the implicit type the starting point of analysis cannot be the Navier-Stokes equations but Cauchy's equation of balance of linear momentum

$$\rho \frac{dv}{dt} = \text{div } \mathbf{T} + \mathbf{b} \quad (44)$$

where \mathbf{b} is the body force.

Models of the type (41) have a much richer class of solutions than the Navier-Stokes model (39). Even in the case of flow between infinite parallel plates, non-unique solutions arise that have no counterparts in the Navier-Stokes theory. In addition, the structure of the solutions to (41) can qualitatively differ from those for the Navier-Stokes fluid (see [30]). In the case of simple Poiseuille flow, the solution may vary from that of plug flow to a V shaped profile, while it is parabolic for the Navier-Stokes fluid.

As the model (41) leads to solutions that qualitatively differ from the solutions to the Navier-Stokes equations, it is also possible that approximations derived from it could lead to equations that predict response that is not only quantitatively but also qualitatively different from the classical approximation. The question here is the magnitude of the error made by not acknowledging the dependence of the viscosity on the pressure consistently while deriving the equations governing the problem of elastohydrodynamics.

3.2. The piezoviscous fluid

Ever since Stokes assumed that the viscosity is a constant, it has been treated as such in most, though not all, subsequent studies. Of course, there are many fluids that shear-thin or shear-thicken; for such fluids, the viscosity is considered a function of the symmetric part of the velocity gradient. However, in one area of research that presupposes the fluid to be a Navier-Stokes fluid, the viscosity is not treated as a constant but is allowed to depend on the pressure, namely in elastohydrodynamic lubrication. Here, we come across a rather intriguing inconsistency. The Reynolds approximation is based on the assumption of constant viscosity. However, in developing the elastohydrodynamic approximation, pressure dependence of the viscosity is acknowledged only after the equation has been obtained under the assumption of constant viscosity [31]. It is astonishing that this obvious inconsistency in the derivation of the equation for elastohydrodynamic lubrication has gone unnoticed until quite recently.

Current research on piezoviscous fluids has raised questions concerning (1) the appropriateness of the Reynolds equation in elastohydrodynamic lubrication (EHL), because of possible change of type of the equations of motion at high pressures, and (2) the errors inherent in the lubrication approximation due to potential existence of cross-film pressure gradient.

There have been some rigorous studies concerning the existence of solutions to the equations governing the flows of fluids with pressure dependent viscosity. Renardy [32] recognized that the equations could change type if the class of viscosity functions that he picked did not satisfy a certain condition. However, his choice of the viscosity functions is unrealistic, as he demands that it satisfy

$\mu(p)/p \rightarrow 0$, as $p \rightarrow \infty$. Numerous experimental results clearly contradict this assumption [33-36]. Gazzola [37] and Gazzola and Secchi [38] have proved local-in-time existence of solutions to the flow of fluids with pressure dependent viscosities. More recently, Malek *et al.* [28, 29] have established existence of solutions that are global-in-time when the viscosity depends on the pressure and the shear rate. However, these results were proven only for spatially periodic flows. Existence of solutions that are global-in-time for the standard Dirichlet problem when $\mu(p)/p \rightarrow \infty$, as $p \rightarrow \infty$ is an open problem.

Bair *et al.* [39] picked up on the criterion that Renardy [32] required for the equations to remain elliptic. Adopting the Barus equation, $\mu = \mu_0 \exp(\alpha p)$, $\mu' = \alpha \mu$, to represent the pressure dependence of viscosity, they re-cast Renardy's criterion. According to Bair *et al.*, in two-dimensional flow change of type from elliptic to hyperbolic occurs when $\tau_I = \alpha^{-1}$. Here $\tau_I = 2\mu d_I$ is the principal shear stress. For mineral oils $\alpha^{-1} \approx 50$ MPa, and the criterion sets a limiting value for the principal shear stress. The practical value of this finding to EHL remains questionable, however, as the Barus formula is unrealistic for glass forming liquids. Moreover, in view of Renardy's analysis requiring unrealistic physical conditions, the modifications proposed are not relevant to the flows of lubricants.

Bair *et al.* [39] appear to be the first to argue, "the Reynolds equation adequately captures the mechanics of the piezoviscous liquid only when the shear stress is much less than the reciprocal of the pressure viscosity coefficient." Schafer, *et al.* [40] continued along this line of investigation and, starting from the Navier-Stokes equations with pressure dependent viscosity, derived a corrected Reynolds equation. They concluded, "application of Reynolds equation is permissible for the case of pure rolling in the contact, but not when considering partial or pure sliding." Greenwood [41], in a discussion to Schafer's paper, offered a simpler derivation of the same equation. Referring to Schafer's paper, Greenwood remarks "the author's astonishing claim that the whole EHL theory is based on an incorrect equation, seems to this discussor to be entirely correct." Full Navier-Stokes solutions of the EHL problem applying the Roelands viscosity-pressure relationship have been produced by Almqvist and Larsson [42].

We follow the procedure that is usually employed to derive equation (6), but use equation (44) as our starting point instead of equation (7). On substituting (41) into the balance of linear momentum (44) we obtain [43]

$$-\text{grad } p + \mu(p)\Delta \mathbf{v} + 2\mathbf{D}[\text{grad } \mu(p)] + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \quad (45)$$

We drop the body force \mathbf{b} and restrict our attention to steady two-dimensional plane flows

$$-\frac{\partial p}{\partial x} + \mu(p)\Delta u + 2\mu'(p)\frac{\partial u}{\partial x}\frac{\partial p}{\partial x} + \mu'(p)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial p}{\partial y} = \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right], \quad (46a)$$

$$-\frac{\partial p}{\partial y} + \mu(p)\Delta v + \mu'(p)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial p}{\partial x} + 2\mu'(p)\frac{\partial v}{\partial y}\frac{\partial p}{\partial y} = \rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]. \quad (46b)$$

Equations (46a, b), will now be recast in terms of non-dimensional variables, defined by

$$(X, Y) = \frac{1}{L_{xz}}\left(x, \frac{1}{\varepsilon}y\right), \quad (U, V) = \frac{1}{U_*}\left(u, \frac{1}{\varepsilon}v\right), \quad P = \frac{p}{P_*}, \quad \bar{\mu} = \frac{\mu}{\mu_*} \quad (47)$$

$$Re^* = \frac{\rho U_* L_y}{\mu_*} \varepsilon, \quad \varepsilon = \frac{L_y}{L_{xz}}$$

μ_* and P_* represent characteristic viscosity and pressure, respectively. For expediency, we chose, $\mu_* = \mu_0$, $P_* = \mu_* U_x L_{xz} / L_y^2$ and employ the Barus formula

$$\bar{\mu} = \exp(\bar{\alpha}P), \quad \bar{\alpha} = \alpha P_* \quad (48)$$

with constant coefficient α , to characterize the pressure dependence of the viscosity.

It is not suggested here that the Barus formula (48) has much validity in the context of EHL calculations and is employed here only for illustrative purposes. Better fit to experiments, at least far from glass transition, is provided by Roelands [33], whose formula

$$\bar{\mu} = \exp\{(\ln \mu_0 + 9.67)[-1 + (1 + 5.1 \times 10^{-9} p)^{const}]\}$$

has been employed in EHL calculations by numerous investigators (see e.g., [44]). Recent viscosity measurements indicate, however, that for glass forming liquids the increase of viscosity with pressure is far more severe when nearing glass transition. Paluch *et al.* [34] find that in certain low-molecular-weight liquids viscosity variation with pressure can be described by

$$\bar{\mu} = \exp\left[\left(\frac{\text{const}}{P_0 - p}\right)p\right]$$

in effect replacing the constant exponent, α , of Barus by $[\text{const}/(P_0 - p)]$ where P_0 is the pressure of the ideal glass transition. Based on their experiments, Irving and Barlow [35] recommended a double exponential in the form

$$\bar{\mu} = \exp(Ae^{Bp} - Ce^{-Dp})$$

where A , B , C , and D , are constants at a given temperature.

Using (48), equations (46) are transformed into (here we drop the overhead bar on μ and α)

$$-\frac{\partial P}{\partial X} + \mu\left(\varepsilon^2 \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + \alpha\mu\left[\frac{\partial U}{\partial Y} \frac{\partial P}{\partial Y} + \varepsilon^2\left(\frac{\partial V}{\partial X} \frac{\partial P}{\partial Y} + 2\frac{\partial U}{\partial X} \frac{\partial P}{\partial X}\right)\right] = Re^*\left(U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y}\right) \quad (49a)$$

$$-\frac{1}{\varepsilon^2} \frac{\partial P}{\partial Y} + \mu\left(\varepsilon^2 \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + \alpha\mu\left[\frac{\partial U}{\partial Y} \frac{\partial P}{\partial X} + \varepsilon^2 \frac{\partial V}{\partial X} \frac{\partial P}{\partial X} + 2\frac{\partial V}{\partial Y} \frac{\partial P}{\partial Y}\right] = Re^*\left(U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y}\right) \quad (49b)$$

Let us now examine the consequence of taking $\varepsilon^2 \rightarrow 0$. On close examination of (49), we find that while the non-dimensional velocities and their derivatives are $O(1)$, the same does not hold true for the derivatives of the non-dimensional pressure. We must keep this in mind and neglect only $O(1)$ terms among those that are multiplied by ε^2 . Consistent with classical lubrication theory, we also assume that $Re^* \rightarrow 0$ and obtain

$$-\frac{\partial P}{\partial X} + \mu\left\{\frac{\partial^2 U}{\partial Y^2} + \alpha\left[\frac{\partial U}{\partial Y} \frac{\partial P}{\partial Y} + \varepsilon^2\left(\frac{\partial V}{\partial X} \frac{\partial P}{\partial Y} + 2\frac{\partial U}{\partial X} \frac{\partial P}{\partial X}\right)\right]\right\} = 0, \quad (50a)$$

$$-\varepsilon^{-2} \frac{\partial P}{\partial Y} + \mu\left\{\frac{\partial^2 V}{\partial Y^2} + \alpha\left[\frac{\partial U}{\partial Y} \frac{\partial P}{\partial X} + \varepsilon^2 \frac{\partial V}{\partial X} \frac{\partial P}{\partial X} + 2\frac{\partial V}{\partial Y} \frac{\partial P}{\partial Y}\right]\right\} = 0. \quad (50b)$$

We now appeal to equation (50) to estimate the order of magnitude of the pressure derivatives. To do this, we require two assumptions, the first of which has already been employed in arriving at equation (50).

Assumptions

(i) All velocities and their derivatives are $O(1)$, i.e., in (50) we have

$$\frac{\partial U}{\partial Y} \frac{\partial P}{\partial X} \gg \varepsilon^2 \frac{\partial V}{\partial X} \frac{\partial P}{\partial X}.$$

(ii) The pressure derivatives are not of the same order, in fact

$$\begin{aligned} \frac{\partial P}{\partial X} &\gg \frac{\partial P}{\partial Y}, \\ \alpha \frac{\partial U}{\partial Y} \frac{\partial P}{\partial X} &\gg \frac{\partial^2 V}{\partial Y^2}. \end{aligned}$$

On employing (i) and (ii) above, in conjunction with equation (50), we obtain the relative order of magnitude of the pressure derivatives

$$\frac{\partial P}{\partial Y} = \alpha \mu \varepsilon^2 \frac{\partial P}{\partial X} \frac{\partial U}{\partial Y}.$$

It follows immediately from assumptions (i) and (ii) that (now in primitive variables)

$$\begin{aligned} \frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial y^2} + \frac{d\mu}{dp} \left[\frac{\partial u}{\partial y} \frac{\partial p}{\partial y} + 2 \frac{\partial u}{\partial x} \frac{\partial p}{\partial x} \right], \\ \frac{\partial p}{\partial y} &= \frac{d\mu}{dp} \frac{\partial p}{\partial x} \frac{\partial u}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{51}$$

To first approximation, flow of a lubricant with pressure dependent viscosity is governed by the system of equations (51). This system contains three equations in three unknowns and, presumably, can be solved to provide a more accurate estimate of the consequences of ignoring the pressure dependence of the viscosity. Nevertheless, here we make the additional simplifying assumption of neglecting the second of equations (51), merely to render the system more amenable to computations, as we demonstrate the effect of the additional terms. Under the approximations made here, the system of equations (51) reduces to

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} + 2 \frac{d\mu}{dp} \frac{\partial u}{\partial x} \frac{dp}{dx} , \quad (52)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \quad (53)$$

Following the analysis of Reynolds, we formally integrate equation (52) subject to the usual boundary conditions and substitute the result into equation (53). Upon integration across the film and observing that $v(h) = v(0) = 0$, we obtain

$$\frac{d}{dx} \left[\left(\frac{h^3}{\mu} - 12\alpha \int_0^h y(h-y) \frac{\partial u}{\partial x} dy \right) \frac{dp}{dx} \right] = 6\tilde{U} \frac{dh}{dx} . \quad (54)$$

To illustrate the kind of difference that might arise on using the modified equation (54), compared to conventional analysis, we examine the elastic cylinder rolling on a plane under the following conditions

$$\frac{w}{E'R} = 1.05 \times 10^{-5} , \quad \frac{\mu_0 \tilde{U}}{E'R} = 1.0 \times 10^{-11} , \quad \alpha E' = 3.0 \times 10^3$$

$$X_{in} = -3.0 , \quad X_{out} = 1.5 ,$$

Figure 6 displays the pressure distribution and figure 7 the viscosity distribution for the classical Reynolds equation and the modified Reynolds equation, within the vicinity of the pressure peak. As may be concluded, these figures show slightly increased peak pressure and significantly higher peak viscosity for the modified equation, relative to the classical solution. This result is in line with our earlier findings for rigid cylinders.

Numerical solutions of the modified pressure equation show that proper accounting for pressure dependence of the lubricants viscosity yields slightly higher pressures, but at much increased viscosity, relative to classical analysis. This conclusion holds for both rigid and elastic cylinders in a cylinder-rolling-on-a-plane problem.

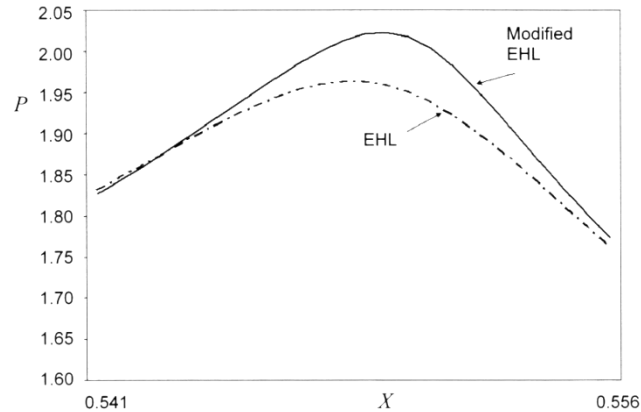


Figure 6. Non-dimensional pressure.

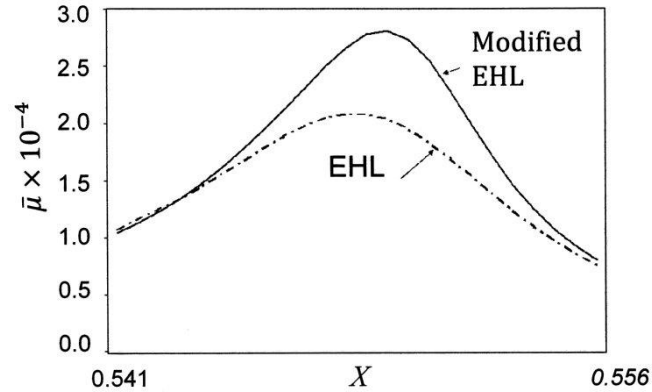


Figure 7. Non-dimensional viscosity.

4. Molecularly thin films

Although the lubrication approximation is derived for thin films, there is, nevertheless, a thin film limit to the approximation's validity. When the characteristic dimensions of the device containing the fluid approach the mean free path (for gases) or the dimension of the molecules (for liquids) the continuum assumption breaks down. There are two distinct representations at our disposal for fluids, continuum and particle. The former is applicable only with restrictions while the particle representation is valid over the whole range of conditions.

The mathematical models that specify the representation of molecular interactions in fluids are shown in table 3 [45]. Particle based representation, which is at the most fundamental level of this hierarchy, is of two kinds, deterministic, in

which the motion of each particle in an ensemble of molecules is followed, and statistical, in which the evolution of the probability density function for the molecules is investigated [46].

Table 3. Types of analysis.

	Type	Equation (Method)	Fluid
Continuum	Deterministic	Navier-Stokes, no-slip (CFD)	Gas-liquid
		Navier-Stokes, slip (CFD)	Gas-liquid
Particle	Deterministic	Newton (MD)	Liquid
	Statistical	Liouville (DSMC)	Gas
		Boltzmann (CFD)	Gas

The deterministic particle based method, molecular dynamics (MD) simulation [47], although theoretically valid for a whole range of conditions, is employed mainly for liquids, as the long flight paths between collisions for gases makes forward integration of the equations prohibitively expensive. In a liquid, the molecules are densely packed, leading to a more efficient application of MD simulation. Sanbonmatsu and Tung, [48] simulated the dynamics of 2.64×10^6 atoms for a total of 22 ns sampling.

The statistics based particle method, on the other hand, presupposes well developed kinetic theory, which is not available for liquids. In addition, the equations employed here are derived for low-density packing of molecules, making these methods applicable to gases. Both Monte Carlo approaches and the Boltzmann equation are derived from the Liouville equation, a conservation equation of the n-dimensional probability function. The direct simulation Monte Carlo (DSMC) method [49, 50] may be used for dilute fluids when the ratio of average molecular spacing to molecular diameter $\sigma/d \geq 10$. The basic assumption of DSMC is to uncouple molecular motions from intermolecular collisions over small time intervals. Particle motions are modeled deterministically, while collisions are treated statistically. The Boltzmann equation for the one-particle distribution function $f(\mathbf{x}, \mathbf{c}, t)$, where \mathbf{x} is the location of the particle and \mathbf{c} its velocity, is applicable over the whole range of the Knudsen number, $Kn = \lambda/h$; it is usually solved by computational fluid dynamics (CFD) methods.

Breakdown of the continuum model is best illustrated for gas flow, as gases have well-developed kinetic theory.

When the characteristic dimension of the flow device can accommodate a large enough number of gas molecules, the fluid can be considered to have matter continuously distributed throughout the space it occupies. However, as devices are made smaller and smaller, attention must be paid eventually to the fact that the

gases consist of discrete molecules. The generally chaotic motion of these molecules, at speeds comparable to the velocity of sound, is punctuated by frequent collisions of about 10^{10} collisions per second. If the density of the gas is small, i.e., the average spacing of the molecules relative to their dimension is large, the collisions between molecules will be binary collisions.

There are numerous technical applications of gas flow in ultrathin channels. One such application is data recording. The density of information storage on a disk increases dramatically with a decrease of the flying height of the read/write head above the disk. However, the flying height cannot be reduced to zero, as this would cause excessive wear of the surfaces. Currently, computer hard drives are manufactured in which the minimum separation of the read/write head from the disk is of the order of the mean free path.

For air at standard temperature and pressure (STP) the ratio of mean free path λ , average spacing of molecules σ , and molecular diameter d are $\lambda : \sigma : d \sim 170 : 10 : 1$. A cube $1 \mu\text{m}$ at the edges contains $n \sim 2.9 \times 10^7$ molecules, and the mean free path is $\lambda \sim 60 \text{ nm}$. Between collisions, the molecules travel along straight-line trajectories, with no intermolecular forces acting on them [58].

The conditions that apply to a gas in thin films are best described with reference to the Knudsen number, $Kn = \lambda/L_y$ (cf. figure 8). $Kn \rightarrow 0$ is the domain of continuum flow while $Kn \rightarrow \infty$ typifies collisionless molecular flow. The various Knudsen number regimes are: $Kn = 0$ for Euler flow, for $0 < Kn < 0.001$ the flow is governed by the Navier-Stokes equation. For flows with Knudsen number above $Kn = 0.001$, the continuum approach is still usable if we allow slip to occur at the boundaries [45]. This was demonstrated by Shaaf and Sherman [51] among others, who measured the drag on a flat plate in a wind tunnel. The Boltzmann equation holds for the full Kn range, but, as the equation is difficult to solve, the continuum approach is advocated whenever applicable. For $Kn \rightarrow \infty$ the collisionless Boltzmann equation applies.

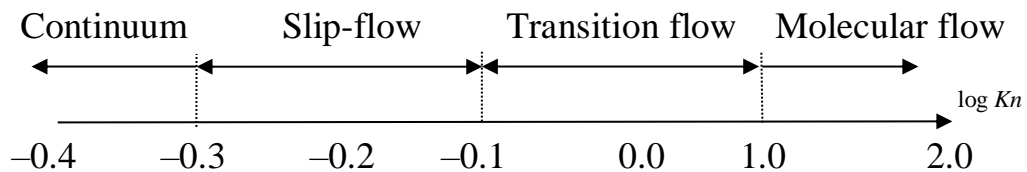


Figure 8. Knudsen number range of various gas flow regimes.

4.1 Velocity slip

It was observed by Kundt and Warburg as early as 1857 that in rarefied gas flow the solid boundary does not support no-slip (velocity) and no-jump (temperature) boundary conditions. A consistent slip condition to make the Navier-Stokes equations valid in the $0.001 < Kn < 0.1$ regime, the so-called slip flow regime, is

$$u_{slip} = L_s \left. \frac{\partial u}{\partial y} \right|_0 \quad (55)$$

where $(\partial u / \partial y)_0$ is the gradient at a point $P \rightarrow wall$. The coefficient L_s is called the slip coefficient; its importance in the slip-flow regime is comparable to the coefficients of viscosity and heat conduction. Albertoni *et al.* [52] tabulated the slip coefficient for various authors, they calculate $L_s = 1.1466\lambda$ from the BGK (Bhatnagar-Gross-Krook) model of the collision term in the Boltzmann equation [53].

To examine the slip-flow boundary conditions we let $u(y)$ represent the velocity parallel to the boundary at y . The shear stress on the small area ds , oriented parallel to the boundary, is given by the momentum transport across it

$$\tau = \nu m (u_+ - u_-), \quad (56)$$

where ν is the frequency of collisions of molecules with ds , m is the molecular mass and $u_+ = u(y^+)$, $u_- = u(y^-)$. The velocities u_+ and u_- are provided by a Taylor expansion of u about y :

$$u_{+,-} = u(y) + \frac{\partial u}{\partial y} (y^{+,-} - y) + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} (y^{+,-} - y)^2. \quad (57)$$

Substituting (57) into (56) we obtain

$$\tau = \nu m \frac{\partial u}{\partial y} (y^+ - y^-) \quad (58)$$

and by comparing (58) with the macroscopic equation $\tau = \mu (\partial u / \partial y)$, the viscosity is found to be

$$\mu = \nu m (y^+ - y^-). \quad (59)$$

Now applying the foregoing to the boundary at y , where y^- is the location of the “displaced” solid wall, so that

$$u_- = u_{plate}, \quad u(y) = u_{gas} \text{ (velocity of gas at wall),}$$

we have

$$u_+ = u_{gas} + \left. \frac{\partial u}{\partial y} \right|_{y=0} (y^+ - y) + \frac{1}{2} \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0} (y^+ - y)^2 \quad (60)$$

Substituting equation (60) into equation (59), and making use of equation (58), we find

$$\begin{aligned} \tau &= \alpha \nu m \left[\left. \frac{\partial u}{\partial y} (y^+ - y) + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} (y^+ - y)^2 + u_{gas} - u_{plate} \right] \\ &= \nu m (y^+ - y^-) \left. \frac{\partial u}{\partial y} \right|_{y=0}. \end{aligned} \quad (61)$$

The slip velocity at the boundary is calculated as the difference between the apparent velocity at the wall, u_{gas} , and the prescribed plate velocity, u_{plate} , [54]

$$u_{slip} = u_{gas} - u_{plate} = \frac{(y^+ - y^-) - \alpha (y^+ - y)}{\alpha} \left. \frac{\partial u}{\partial y} \right|_{y=0} - \frac{1}{2} \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0} (y^+ - y)^2 \quad (62)$$

Here we introduced α , $0 < \alpha < 1$, the momentum accommodation coefficient (see e.g., Bird, 1994).

In the 1st order slip [55] and the 2nd order slip [56] models $y^+ - y = \lambda$, while $y^+ - y = (2/3)\lambda$ in the 1.5 order slip model of [57], taking into account that the y -direction component of the averaged distance between collisions is $(2/3)\lambda$ (see Vincenti and Kruger [58] and Bird [49] on this topic).

Using equation (8) and $U_* = \sqrt{2RT_0}$ to define nondimensional coordinates and velocity components, respectively, and putting $(2 - \alpha)/\alpha = a$

$$\begin{aligned}
 U|_{Y=1} &= U_0 + a Kn \frac{\partial U}{\partial Y} \Big|_n + \frac{1}{2} (\beta Kn)^2 \frac{\partial^2 U}{\partial Y^2} \Big|_n \\
 U|_{Y=0} &= -a Kn \frac{\partial U}{\partial Y} \Big|_0 - \frac{1}{2} (\beta Kn)^2 \frac{\partial^2 U}{\partial Y^2} \Big|_0
 \end{aligned} \tag{63}$$

In the notation of equation (8), $\beta = 0$ for 1st order slip, $\alpha = \beta = 1$ for 2nd order slip, and $\beta = 2/3$ for 1.5 order slip (however, Mitsuya multiplies the first order term by 3/2). Using the boundary conditions (63), the velocity distribution along x is given as

$$u = \frac{h^2}{2\mu} \frac{\partial p}{\partial x} \left[\left(\frac{y}{h} \right)^2 - \left(\frac{y}{h} \right) - a Kn - (\beta Kn)^2 \right] + U \left[1 - \frac{(y/h) + Kn}{1 + 2 Kn} \right] \tag{64}$$

Using equation (64) and a similar expression for w , a Reynolds equation can be derived by methods similar to those for incompressible fluids (see e.g., [2]). The procedure yields

$$\begin{aligned}
 \frac{\partial}{\partial X} \left\{ PH^3 \frac{\partial P}{\partial X} \left[1 + 6a \left(\frac{Kn}{PH} \right) + 6 \left(\beta \frac{Kn}{PH} \right)^2 \right] \right\} \\
 + \frac{\partial}{\partial Z} \left\{ PH^3 \frac{\partial P}{\partial Z} \left[1 + 6a \left(\frac{Kn}{PH} \right) + 6 \left(\beta \frac{Kn}{PH} \right)^2 \right] \right\} = \Lambda \frac{\partial PH}{\partial X}
 \end{aligned} \tag{65}$$

Here p_a is the ambient pressure, h_0 is the minimum film thickness, $P = p/p_a$, $H = h/h_0$ and the bearing number Λ has the definition

$$\Lambda = \frac{6\mu UL}{p_a h_0^2}$$

The Reynolds equation for continuum gas flow can be obtained from equation (65) by setting $\alpha = \beta = 0$.

Wu *et al.* [59] derived first and second order slip models by summing the contributions from each group of molecules impinging on the surface at an angle; this, in effect, means relaxing the requirement that the length scale in the Taylor

expansion (62) equals the mean free path (20). The coefficients multiplying the Kn/PH and the $(Kn/PH)^2$ terms in their scheme are $4a$ and 3 , respectively.

Comparison with experiments suggest that the continuum model allowing for slip at the boundaries yields good results for $Kn \leq 1$ according to Odaka *et al.* (quoted by Fukui and Kaneko [60]), and even for $Kn \leq 2.5$ according to Hsia and Domoto [56]. Nevertheless, it is difficult to justify its use for $Kn > 1$.

4.2. Molecular gas lubrication

To construct a model of gas flow valid for arbitrary Knudsen number, we must make recourse to the Boltzmann equation, a conservation equation of the one-particle distribution function $f(x, y, z, c_x, c_y, c_z, t)$

$$\frac{\partial f}{\partial t} + \mathbf{c} \cdot \nabla f = \left\{ \frac{\delta f}{\delta t} \right\}_{coll} \quad (66)$$

Here, $f(c_i)dV_x dV_c$ is the expected number of molecules that lie simultaneously in the volume elements $dV_x = dx_1 dx_2 dx_3$ of physical space, located at \mathbf{x} , and $dV_c = dc_1 dc_2 dc_3$ of velocity space, located at \mathbf{c} [58]. The right hand side of equation (66) represents the collision integral that contains the probability distribution $f(c_i)$, making equation (66) a non-linear integro-differential equation.

Because of the nonlinearity of the collision term, the Boltzmann equation is not easy to solve. A particular class of solutions, namely the Maxwellians, describes equilibrium states. The left hand side of the Boltzmann equation (66) is then zero and the equilibrium distribution function, f_0 , is its solution [58].

$$\left\{ \frac{\delta f_0}{\delta t} \right\}_{coll} = 0$$

The Maxwellian distribution function is given by

$$f_0 = n_0 \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left[\frac{-m(\mathbf{c} - \mathbf{v})^2}{2kT} \right], \quad (67)$$

where k is the Boltzmann constant, and T is the temperature.

Expanding the Boltzmann equation in powers of the Knudsen number [61]

$$f = f_0 + Kn f_1 + Kn^2 f_2 + \dots,$$

the first term is the Maxwellian equilibrium distribution and the corresponding conservation equation simplifies to Euler's equation of inviscid flow. The second equation reduces to the Navier-Stokes equation while the third term produces the Burnett equation.

For small departures from equilibrium, when the macroscopic velocity of the gas is small relative to the thermal velocity of its molecules, the collision integral can be linearized [60, 62]. Note that since the velocity of sound is of the order of the root mean square of the molecular velocity, the condition of linearity may be written in terms of the Mach number as $M \rightarrow 0$.

To derive a Reynolds type equation from the linearized form of equation (66) the flow rate must first be calculated [60]. Due to linearity, the flow rate q is a superposition of the pressure flow q_p , the Couette flow q_C and the thermal creep flow q_T ,

$$q = q_p + q_C + q_T$$

Owing to the symmetry property of the Couette velocity profile, q_C is not dependent on the Knudsen number. As for the other two flow components, q_p has been evaluated by Cercignani *et al.* [63, 64]

$$q_p = -Q_p \frac{h^2}{\sqrt{2RT_0}} \left(\frac{dp}{dx} \right) \quad (68a)$$

and q_T by Loyalka [65]

$$q_T = Q_T \frac{ph^2}{T_0 \sqrt{2RT_0}} \left(\frac{dT_w}{dx} \right). \quad (68b)$$

In equation (68) the Q_p and Q_T are functions of the Knudsen number, they can be found in the above cited papers and also reproduced in Fukui and Kaneko [60].

A generalized Reynolds equation based on the Boltzmann equation can now be expressed as [60]

$$\text{div} \left\{ PH^3 \left[\overline{Q}_p(D_0PH, \alpha) \cdot \text{grad}P - \overline{Q}_T(D_0PH, \alpha) \cdot P \cdot \text{grad}\tau_w \right] \right\} = \Lambda \text{grad}(PH), \quad (69)$$

where $D_0 = p_0 h_0 / \mu \sqrt{2RT}$ is the “characteristic” inverse Knudsen number calculated on the reference state, and is an assigned parameter.

The problem, as presented by equation (69) is laborious to solve. The nonlinear coefficients must be obtained from certain integral equations at each step of the iteration. Fukui and Kaneko presented an alternative method of solution in a later paper [66], in which they solved the finite difference approximation to the integral equations referred to above, ahead of time then used the newly created database to interpolate for flow coefficients in terms of the Knudsen number.

In the next few figures, reproduced from Fukui and Kaneko [60] and Mitsuya [57], we shall compare results from the various models with one another and with experimental data. The variation of the pressure flow rate, Q_p , with the inverse Knudsen number is shown in figure 9. The continuum theory with no-slip yields acceptable results for $Kn < 0.01$, while the 2nd order slip result is good for $Kn < 1$. For $Kn > 1$, the Boltzmann equation results differ drastically from slip flow model predictions.

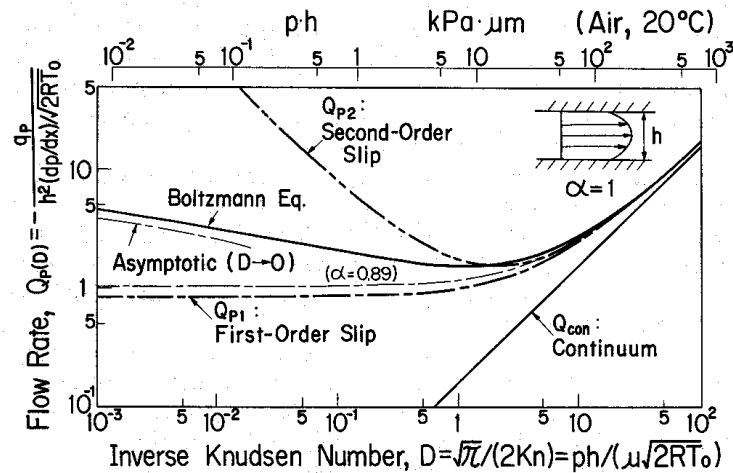


Figure 9. Variation of pressure flow coefficient with the inverse Knudsen number [60].

The load capacity of inclined plates in relative sliding at $A = 10$ and varying D is shown in figure 10. The Boltzmann solution seems to be bracketed by the 1st order slip and 2nd order slip solutions.

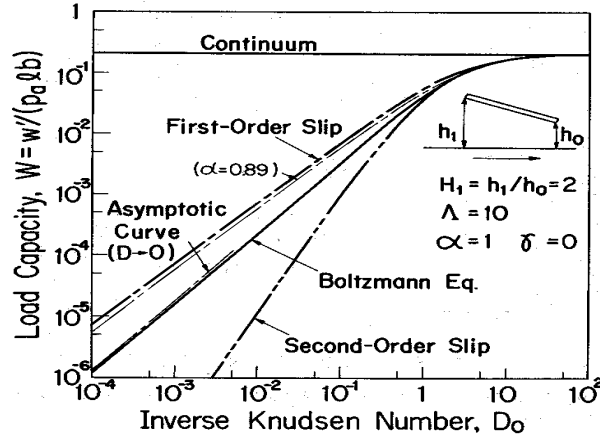


Figure 10. Load capacity of plane slider [60].

As a final comparison between model predictions and experiment, figure 11 plots minimum film thickness between a read/write head and computer disk. The 1.5 order slip model seems to perform best in this case, after the Boltzmann model, of course.

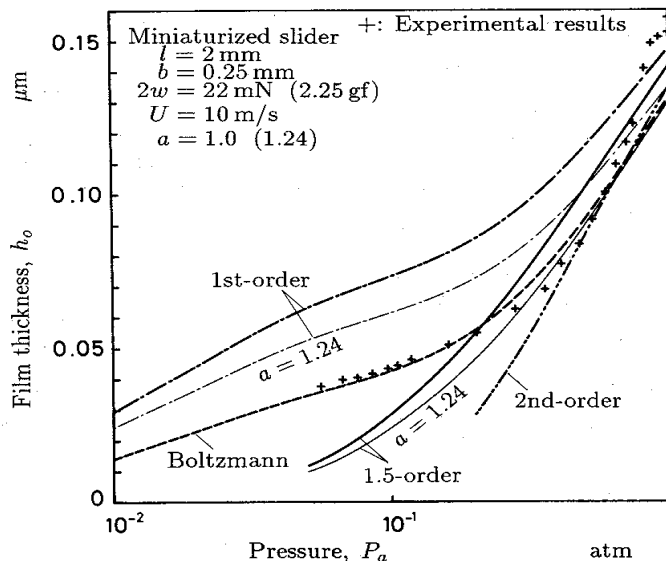


Figure 11. Minimum film thickness for slider [57].

4.3. Liquids

As the temperature of a gas is lowered, the thermal velocity of the molecules decreases and the molecules become more densely packed. A cube of liquid $1 \mu\text{m}$ at the edges now contains $n \sim 3.35 \times 10^{10}$ molecules, and the average molecular spacing decreases to $\sigma = 0.31 \text{ nm}$. We can no longer speak of mean free path, as

the molecules are now too closely packed. Liquid molecules are in a continuous state of collision.

The difference between liquids and gases is that for gases there are no forces acting on a molecule between successive collisions, while the molecules of a liquid are so closely packed that the forces of attraction between molecules can manifest. Therefore, it might be expected that the behavior of liquids and gases close to a solid boundary will also show similarity and under certain surface-liquid conditions the no-slip boundary condition will not be satisfied. The degree of slip will then depend on the strength of the solid-fluid coupling.

Experimental evidence is contradictory in this respect. Derjaguin (quoted in [67]) found that the viscosity of water in quartz capillaries with a diameter less than 100 nm, attains a value 40-50% higher than in bulk. However, this conclusion did not hold for non-polar liquids. Chan and Horn [68] studied the drainage of fluids between two atomically smooth mica surfaces. Their results are in excellent agreement with the Reynolds theory of lubrication for film thickness $h > 50$ nm. For thinner films, they find that drainage is somewhat slower than predicted by continuum theory, as about two molecular layers on each surface undergo no shear. Thus, in thinner films there is an apparent enhancement of viscosity, which can be accounted for by allowing the plane of shear to be displaced into the liquid. Israelachvili [67] reported that in films as thin as 5 nm the “plane of slip” is within a few Angstrom units of the interface and the viscosity is within 10% of its bulk value. Viscosity increase in thin channels was reported by Migun and Prokhorenko [69], while Debye and Cleland [70], and Pfahler *et al.* [71] found the apparent viscosity, μ_a , consistently smaller than μ .

Liquids in large gaps, or liquids above a single wall, are fluid all the way to within one or two molecular layers of the solid surfaces. When the gap is squeezed down to the thickness of a few molecules, the confined liquid is solid-like. Klein and Kumacheva [72] find that as the film is made increasingly thinner, the effective viscosity of the liquid changes by at least seven orders of magnitude over a change in film thickness of a single molecular spacing (figure 12)². When $n > n_c$, where n represents the number of molecules across the film and n_c is the critical number of molecules for liquid-solid transition, there is liquid like behavior, and solid-like behavior when $n < n_c$. In their experiments Klein and Kumacheva [72] find $n_c = 7$.

² Reprinted from Klein, J. and Kumacheva, E., *Physica A*, 249, 206-215 (1998) with permission by Elsevier.

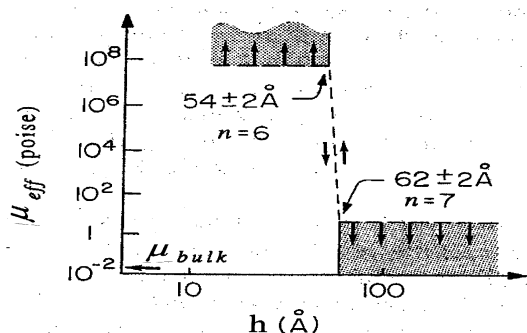


Figure 12. Variation of the effective mean viscosity, μ_{eff} , with film thickness, h . Confined liquid film [72].

This fluid-solid like transition can be attributed to a change in the geometric packing of the molecules, as they undergo layering, extending a few molecules away from the surface. The dramatic increase in viscosity is due to abrupt transition caused by confinement alone. Hu and Granick [73] recognize a special lubrication regime, located between EHL and boundary lubrication. In this thin film regime, “lubricant flow and fluid dynamics are still in action but behave differently from expectations of classical theory.”

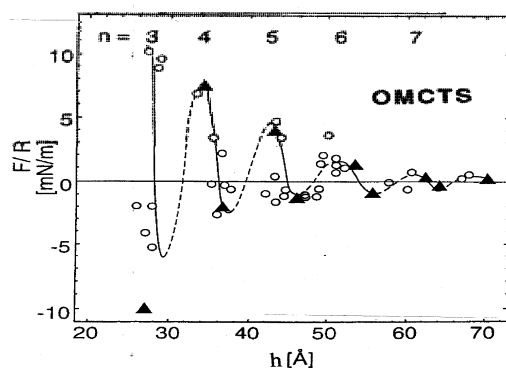


Figure 13. Solvation force, indicating the number n of molecular layers corresponding to each ‘hump’ [72].

When atomically smooth mica sheets are made to approach one another, the normal force acting at large separations is the continuum van der Waals forces. As the separation between the surfaces is decreased, the solvation force becomes an oscillating function of the separation [67, 68], due to the layering of the molecules (deviations from the predictions of continuum theory are often referred to as solvation effects). The period of oscillation is roughly the same as the diameter of the liquid molecules. The number of layers, n , can be counted by dividing the spacing at the maxima by the molecular diameter, as indicated in figure 13².

Jang and Tichy [74] used the exponential-cosine curve fit of Chan and Horn [68]

$$F_{solv} = -RB \exp\left(-\frac{h}{d}\right) \cos\left(\frac{2\pi h}{d}\right), \quad B = 172 \text{ MPa}$$

to represent the solvation pressure in their thin film lubrication (TFL) correction to EHL theory. The film thickness begins to deviate from the conventional EHL theory [75] when the separation is less than 7-8 nm, and changes stepwise. Matsuoka and Kato [76], who calculated the solvation pressure from a force potential [77], in figure 14 compare experimental data with theoretical predictions.

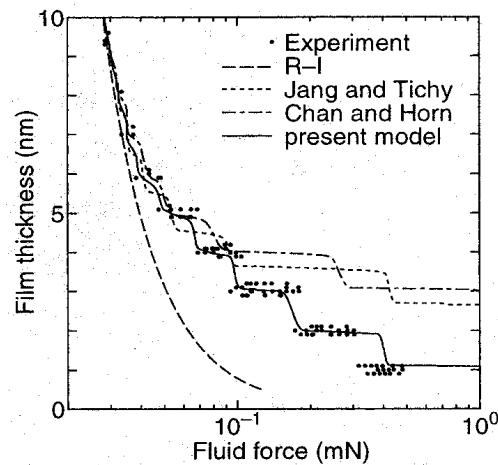


Figure 14. Comparison of three solvation pressure models with experiment for OMCTS [76].

5. Summary

Often, it is not too difficult to state that the just-derived approximation becomes increasingly accurate as a defining parameter approaches a definite value, say zero. It is considerably more challenging but at the same time, more useful, to state with certainty that just how small the parameter must be for the approximation to yield useful results. Nevertheless, we will now attempt to define the range of validity of the major assumptions of Reynolds' lubrication theory.

The continuum assumption with no-slip boundary conditions is valid for films whose thickness is considerably greater than the mean free path for gases or significantly more than 10-12 molecular thickness for liquids. The assumption will

still yield acceptable answers for films whose thickness is comparable to the mean free path, if velocity slip is specified at the boundaries. Liquid films are far less understood than gas films; nevertheless, it is clear that for confined liquids there is complete agreement with the Navier–Stokes theory, including the no-slip boundary condition except under conditions of extraordinarily high rates of shear, when the film is more than 10 molecules thick. However, a liquid-solid transition occurs when the number of molecules across the film falls below a critical value, the transition resulting in an increase of viscosity by several orders of magnitude. This sequence of events points to the existence of a thin-film lubrication regime, located between EHL and boundary lubrication.

Lubricant inertia is negligible and the classical Reynolds equation yields good results when the reduced Reynolds number $Re^* \leq 1$ and the local value of the film slope $\varepsilon \leq 0.1$. If the reduced Reynolds number is larger than this value but the local value of the slope remains small, the flow is essentially two-dimensional. The pressure, however, must be calculated in such cases from equations that retain fluid inertia; the classical Reynolds equation is no longer valid. The same holds true whether the flow is laminar or turbulent.

Though the derivation of the Reynolds equation currently in use for elastohydrodynamic lubrication is less than rigorous, current EHL theory yields acceptable results in most applications [78]. There might be cases, however, especially those associated with significant heat generation in the film, where the governing equations must be derived from first principles.

6. References

1. Reynolds, O. 1986. On the theory of lubrication and its application to Mr. Beuchamp Tower's experiments. *Phil Trans. Roy. Soc.*, **177**, 157-234.
2. Szeri, A. Z. 1998. *Fluid Film Lubrication: Theory and Design*, Cambridge University Press.
3. Dai, R. X., Dong, Q. M. and Szeri, A. Z. 1992. Approximations in Hydrodynamic Lubrication. *J. Tribology*, **114**, 14-25.
4. Szeri, A. Z. and Snyder, V. 2006. Convective inertia effects in wall-bounded thin film flows. *Meccanica*, **41**, 473-482.
5. Sun, D. C. and Chen, K. K. 1977. First effects of Stokes roughness on hydrodynamic Lubrication. *J. Lub. Tech.*, **99**, 2-9.
6. Van Odyck, D. E. A. and Venner, C. H. 2003. Compressible Stokes flow in thin films. *J. Tribology*, **125**, 543-551.
7. Hinze, J. O. 1975. *Turbulence*, 2d ed., McGraw-Hill, New York.
8. Constantinescu, V. N. 1962. Analysis of bearings operating in turbulent regime. *Trans. ASME*, **82**, 139-151.

9. Elrod, H. G. and Ng, C. W. 1967. A theory for turbulent films and its application to bearings. *Trans. ASME*, **89**, 347-362.
10. Ng, C. W. and Pan, C. H. T. 1965. A linearized turbulent Lubrication theory, *Trans. ASME*, **87**, 675-688.
11. Constantinescu, V. N. 1959. On turbulent lubrication. *Proc. Inst. Mech. Eng.*, **173**, 881-900.
12. de Boor, C. 1978. *A Practical Guide to Splines*. Springer-Verlag.
13. Ritchie, G. S. 1968. On the stability of viscous flow between eccentric rotating cylinders. *J. Fluid Mech.*, **32**, 131-144.
14. Ballal, B. and Rivlin, R. S. 1976. Flow of a Newtonian fluid between eccentric rotating cylinders. *Arch. Rat. Mech. Anal.*, **62**, 237-274.
15. Myllerup, C. M. and Hamrock, B. J. 1994. Perturbation approach to hydrodynamic lubrication theory. *J. Tribology*, **116**, 110-118.
16. Nazarov, S. A. and Videman, J. H. 2007. A modified nonlinear Reynolds equation for thin viscous flows in lubrication. *Asymptotic Analysis*, **52**, 1-36.
17. Becker, J. and Grun, G. 2005. The thin film equation: recent advances and some new perspectives. *J. Phys. Cond. Matter*, **17**, 291-307.
18. Schwartz, L., Roy, R. V., Elly R. R. and Princen H. M. 2004. "Surfactant-driven motion and splitting of droplets on a substrate" *J. Eng. Math.*, **50**, 157-175.
19. Alvarez, A. and Soto, R. 2005. Dynamics of a suspension confined in a thin cell. *Phys. Fluids* **17**, 093103.
20. Munch, A. and Wagner, B. 2005. Contact-line instability of dewetting thin films. *Physica D – Nonlin. Phenomena* **209**, 178-190.
21. Stokes, G. G. 1845. On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Trans. Cambridge Phil. Soc.* **8**, 287-305.
22. Bridgman, P. W. 1931. *The Physics of High-Pressure*, New York, MacMillan.
23. Cutler, W. G., McMickles, R. J., Webb, W. and Schiessler, R. W. 1958. Study of the compressions of several high molecular weight hydrocarbons. *J. Chem. Phys.* **29**, 727-740.
24. Griest, E. M., Webb, W. and Schiessler, R. W. 1958. Effect of pressure on viscosity of higher hydrocarbons and their mixtures, *J. Chem. Phys.*, **29**, 711-720.
25. Johnson, K. L. and Cameron, R. 1967. Shear behaviour of elastohydrodynamic oil films at high rolling contact pressures. *Proc. Instn. Mech. Engrs.*, **182**, 307-319.
26. Johnson, K. L. and Greenwood, J. A. 1980. Thermal analysis of an Eyring fluid in elastohydro-dynamic traction. *Wear* **61**, 355-374.
27. Johnson, K. L. and Tevaarwerk, J. L. 1977. Shear behaviour of elastohydrodynamic oil films. *Proc. R. Soc. Lond.*, **A356**, 215-236.
28. Malek, J., Necas, J. and Rajagopal, K. R. 2002. Global analysis of the flow of fluids with pressure dependent viscosities. *Arch. Ratl. Mech. Anal.*, **165**, 243-269.
29. Malek, J., Necas, J. and Rajagopal, K. R. 2002. Global existence of solutions for flows of fluids with pressure and shear dependent viscosities. *Applied Math. Letters*, **15**, 961-967.
30. Hron, J., Malek, J. and Rajagopal, K.R. 2001. Simple flows of fluids with pressure-dependent viscosities. *Proc. Royal Society Lond.*, **A457**, 1603-1622.
31. Dowson, D. and Higginson, G. R. 1966. *Elastohydrodynamic Lubrication. The Fundamentals of Roller and Gear Lubrication*, Pergamon, Oxford.

32. Renardy, M. 1986. Some remarks on the Navier-Stokes equations with a pressure-dependent viscosity. *Comm. Partial Diff. equation*, **11**, 779-793.
33. Roelands, C. J. A. 1966. Correlation aspects of the viscosity-temperature-pressure relationship of lubricating oils. PhD dissertation, Technische Hogeschool Delft, The Netherlands.
34. Paluch, M., Dzendzik, Z. and Rzożka, S. J. 1999. Scaling of high -pressure viscosity data in low-molecular-weight glass-forming liquids. *Phys. Rev. B*, **60**, 2979-2982.
35. Irving J. B. and Barlow, A. J. 1971. An automatic high pressure viscometer. *J. Phys. E.*, **4**, 233-236.
36. Bendler, J. T., Fontanella, J. J. and Schlesinger, M. F. 2001. A new Vogel-like law: ionic conductivity, dielectric relaxation, and viscosity near the glass transition. *Phys. Rev. Let.*, **87**, 195503-1-4.
37. Gazzola, F. 1997. A note on the solution of Navier-Stokes equations with a pressure dependent viscosity. *Z. Angew. Math. Phys.* **48**, 760-773.
38. Gazzola, F. and Secchi, P. 1998. Some results about stationary Navier-Stokes equations with a pressure dependent viscosity. In *Navier-Stokes Equations: Theory and Numerical Methods* (ed. R. Salvi), 31-37, New York, Longman.
39. Bair, S., Khonsari, M. and Winer, W. O. 1998. High-pressure rheology of lubricants and limitations of the Reynolds equation. *Trib. Int.*, **10**, 573-586.
40. Schafer, C. T., Giese, P., Rowe, W. B. and Woolley, N. H. 1999. Elastohydrodynamically lubricated line contact based on the Navier-Stokes equations. *Proc. Leeds-Lyon Symp. Trib.*, 57-68, Elsevier. *Soc.*, **177**, 157-234.
41. Greenwood, J. A. 2000. In *Thinning Films and Tribological Interfaces. Proc 26th Leeds-Lyon Symp.* (ed. D. Dowson), Tribology series, **28**, 793-794.
42. Almqvist, T. and Larsson, R. 2002. The Navier-Stokes approach for thermal EHL line contact solution. *Trib. Int.*, **35**, 163-170.
43. Rajagopal, K. R. and Szeri, A. Z. 2003. On an inconsistency in the derivation of the equations of elastohydrodynamic lubrication. *Proc. Roy. Soc. Lond.*, **A 459**, 2771-2786.
44. Venner, C. H. and Lubrecht, A. A. 1994. Transient analysis of surface features in an EHL line contact in the case of sliding. *J. Tribology*, **116**, 186-193.
45. Gad-el-Hak, M. 1999. The fluid mechanics of micro devices - The Freeman Scholar Lecture. *J. Fluids Eng.*, **121**, 5-33.
46. Ungerer, P., Nieto-Draghi, C., Rousseau, B., Ahunbay, G. and Lachet, V. 2007. Molecular simulation of the thermophysical properties of fluids: From understanding toward quantitative predictions. *J. Molecular Liquids*, **134**, 71-89.
47. Koplik, J. and Banavar, J. R. 1995. Continuum Deductions from Molecular Hydrodynamics. *Ann. Rev. Fluid Mech.*, **27**, 957-992.
48. Sanbonmatsu, K.Y and Tung, C.S. 2006. High performance computing in biology: Multimillion atom simulations of nanoscale systems. *J. Structural Biology*, **157**, 470-480.
49. Bird, G. A. 1994. *Molecular Gas Dynamics and the Direct Simulation Of Gas Flows*, Clarendon Press, Oxford.
50. Oran, E. S., Oh, C. K. and Cybyk, B. Z. 1998. Direct Simulation Monte Carlo: Recent Advances and Applications. *Annual Rev. Fluid Mech.*, **30**, 403-441.
51. Schaaf, S. A. and Sherman, F. S. 1954. Skin friction in slip flow. *J. Aero. Sci.*, **21**, 85-90.

52. Albertoni, S., Cercignani, C. and Gotusso, L. 1963. Numerical evaluation of the slip coefficient. *Phys. Fluids*, **6**, 993-996.
53. Bhatnagar, P. L., Gross, E. P. and Krook, M. 1954. A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems. *Phys. Review*, **94**, 511-525.
54. Szeri, A. Z. 2001. Flow modeling of thin films from macroscale to nanoscale. *Fundamentals of Tribology and Bridging the Gap between the Macro-Micro/Nanoscales*, 767-798, Kluwer, Netherlands.
55. Burgdorfer, A. 1959. The influence of molecular mean free path on the performance of hydrodynamic gas lubricated bearings. *J. Basic Engr.*, **81**, 94-100.
56. Hsia, Y. T. and Domoto, G. A. 1983. An experimental investigation of molecular rarefaction effects in gas lubricated bearings at ultra-low clearances. *J. Tribology*, **105**, 120-130.
57. Mitsuya, Y. 1993. Modified Reynolds equation for ultra-thin film gas lubrication using 1.5 order slip flow model and considering surface accommodation coefficient. *J. Tribology*, **115**, 289-294.
58. Vincenti, W. G. and Kruger, C. H. 1965. *Introduction To Physical Gas Dynamics*, John Wiley & Sons, New York.
59. Wu, Lin and Bogy, D. B. 2003. New first and second order slip models for the compressible Reynolds equation. *J. Tribology*, **125**, 558-561.
60. Fukui, S. and Kaneko, R. 1988. Analysis of ultra-thin gas film lubrication based on linearized Boltzmann equation. *J. Tribology*, **110**, 253-261.
61. Kogan, M. N. 1969. *Rarefied Gas Dynamics*, Plenum Press, New York.
62. Gross, F. P., Jackson, E. A. and Ziering, S. 1957. Boundary value problems in kinetic theory of gases. *Ann. of Physics*, **1**, 141-167.
63. Cercignani, C. and Daneri, A. 1963. Flow of a rarefied gas between two parallel plates. *J. Appl. Phys.*, **34**, 3509-3513.
64. Cercignani, C. and Pagani, C. D. 1966. *Phys. Fluids*, **6**, 1167-1175.
65. Loyalka, S. K. 1971. Kinetic theory of thermal transpiration and mechanocaloric effect. *I. J. Chem. Phys.*, **55**, 4497-4503.
66. Fukui, S. and Kaneko, R. 1990. A database for interpolation of Poiseuille flow rates for high Knudsen number lubrication problems. *J. Tribology*, **112**, 78-83.
67. Israelachvili, J. N. 1986. Measurement of the viscosity of liquids in very thin films. *J. Coll. Interface Sci.*, **110**, 263-271.
68. Chan, D. Y. C. and Horn, R. G. 1985. The drainage of thin liquid films between solid surfaces. *J. Chem. Phys.*, **83**, 5311-5324.
69. Migun, N. P. and Prokhorenko, P. P. 1987. Measurement of the viscosity of polar liquids in micro-capillaries. *Colloid J. USSR*, 849-897.
70. Debye, P. and Cleland, R. L. 1959. Flow of liquid hydrocarbons in porous vycor. *J. Appl. Phys.*, **30**, 843-849.
71. Pfahler, J., Harley, J., Bau, H. and Zemel, J. N. 1991. Gas and liquid flow in small channels. *Symposium on Micromechanical Sensors, Actuators and Systems*, ed. D. Cho *et al.* ASME DSC, **32**, 49-60.
72. Klein, J. and Kumacheva, E. 1998. Liquid-to-solid transition in thin liquid films induced by confinement. *Physica A*, **249**, 206-215.

73. Hu, YZ. and Granick, S. 1998. Microscopic study of thin film lubrication and its contribution to macroscopic tribology. *Tribology Letters*, **5**, 81-88.
74. Jang, S. and Tichy, J. A. 1995. Rheological models for thin film EHL contacts. *J. Tribology*, **117**, 22-28.
75. Luo, J., Huang, P. and Li, L. K. Y. 1999. Characteristics of liquid lubricant films at the nano-scale. *ASME J. Trib.*, **121**, 872-878.
76. Matsuoka H. and Kato, T. 1997. An ultrathin liquid film lubrication theory – Calculation method of solvation pressure and its application to the EHL problem. *ASME J. Tribology*, **119**, 217-226.
77. Henderson, D. and Lozada-Cassou, M. 1986. A simple theory for the force between spheres immersed in liquid. *J. Colloid and Interface Sci.*, **114**, 180-183.
78. Gohar, R. 2001. *Elastohydrodynamic Lubrication*. 2nd edition, Imperial College Press.

Copyright

- **Figures 1, 2, 3, 4, 5:** reprinted with permission from Szeri, A. Z. and Snyder, V. Convective inertia effects in wall-bounded thin film flows. *Meccanica*, 2006, **41**, 473-482.
- **Figures 6, 7:** reprinted from Rajagopal, K. R., and Szeri, A. Z. On an inconsistency in the derivation of the equations of elastohydrodynamic lubrication. *Proc. Roy. Soc. Lond. A*, 2003, **459**, 2771-2786.
- **Figures 9, 10:** reprinted with permission from Fukui, S., and Kaneko, R. Analysis of ultra-thin gas film lubrication based on linearized Boltzman equation. *ASME J. Tribology*, 1988, **110**, 253-262.
- **Figure 11:** reprinted with permission from Mitsuya, Y. Modified Reynolds equation for ultra-thin film gas lubrication using 1.5 order slip flow model and considering surface accommodation coefficient. *ASME J. Tribology*, 1993, **115**, 289-294.
- **Figures 12, 13:** reprinted from *Physica A* **249**. Klein, J. and Kumacheva, E. Liquid-to-solid transition in thin liquid films induced by confinement. 206-215, copyright (1998), with permission from Elsevier.
- **Figure 14:** reprinted with permission from Matsuoka, H., and Kato, T. An ultrathin liquid film lubrication theory – calculation method of solvation pressure and its application to the EHL problem. *ASME J. Tribology*, 1997, **119**, 217-226.